§4 Continuity

4.1 Definition

Definition 4.1.1

Let $c \in A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$ be a function.

A function $f(x)$ is said to be continuous at $x = c$ if $\lim_{x \to c} f(x) = f(c)$.

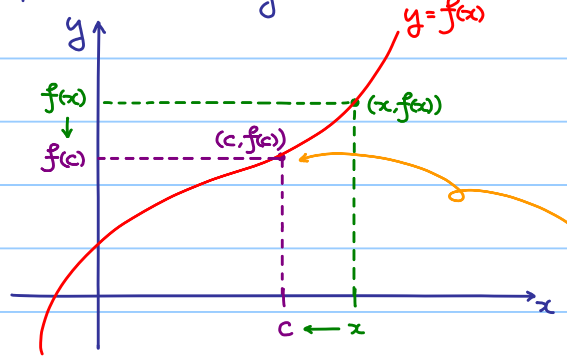
Idea:

③ they equal

$$\lim_{x \to c} f(x) = f(c)$$

① limit exists ② well-defined

Geometrical meaning:



the curve does NOT break up at the point $x = c$!

Furthermore, if a function $f: A \to \mathbb{R}$ is continuous at every point in A, then f is said to be continuous on A.

Let $h = x - c$, i.e. $x = c + h$

When x tends to c, h tends to 0.

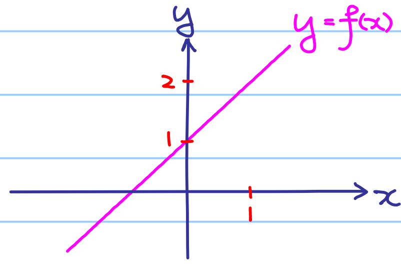
Therefore, we have another formulation:

A function $f(x)$ is said to be continuous at $x = c$ if $\lim_{h \to 0} f(c+h) = f(c)$.

4.2 Examples

Example 4.2.1

Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by $f(x)=x+1$.



We have : ① $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} x+1 = 2$

② $f(1) = (1)+1 = 2$

$\therefore$ f is continuous at $x=1$.

Example 4.2.2

Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ a & \text{if } x = 0 \end{cases}$$

We have $\lim\limits_{\substack{x\to 0 \\ \text{i.e. } x\neq 0}} f(x) = \lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

$\therefore$ f is continuous at $x=0$ unless $\lim\limits_{x\to 0} f(x) = f(0)$, i.e. $a=1$.

Recall :

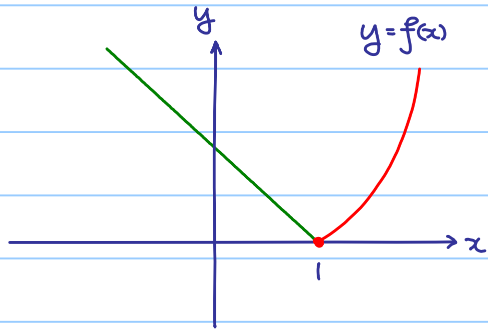
$\lim\limits_{x\to c} f(x) = L$ if and only if $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x) = L$

Rewrite :

A function $f(x)$ is said to be continuous at $x=c$ if $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x) = f(c)$

Example 4.2.3

If $f(x) = \begin{cases} x^2-1 & \text{if } x \geq 1 \\ 1-x & \text{if } x < 1 \end{cases}$



① $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} x^2-1 = 0$

② $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} 1-x = 0$

③ $f(1) = 1^2-1 = 0$

$\therefore$ f is continuous at $x=1$.

Example 4.2.4

Absolute Value : $|x| = \sqrt{x^2}$

For example :

$|3| = \sqrt{3^2} = \sqrt{9} = 3$

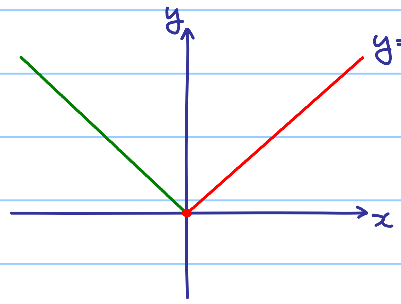
$|0| = \sqrt{0^2} = \sqrt{0} = 0$

$|-3| = \sqrt{(-3)^2} = \sqrt{9} = 3$

(Simply speaking : throw away the negative sign)

Rewrite $|x|$ as a piecewise defined function :

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



We have :

① $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$

② $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -x = 0$

③ $f(0) = 0$

$\therefore$ $|x|$ is continuous at $x = 0$.

Theorem 4.1.1

- If $f(x)$ and $g(x)$ are continuous at $x = c$, then $f(x) \pm g(x)$, $f(x)g(x)$, $\frac{f(x)}{g(x)}$ $(g(c) \neq 0)$ are continuous at $x = c$ as well.
- Polynomial functions and exponential functions are continuous everywhere.
- Trigonometric functions and logarithmic functions are continuous at every point where they are defined.
- If $g(x)$ is continuous at $x = c$ and $f(x)$ is continuous at $x = g(c)$, then $f(g(x))$ is continuous at $x = c$.

(That's why we usually have $\lim_{x \to c} f(x) = f(c)$ as we usually looking at continuous function.)

Example 4.2.5

Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that

(i) f is continuous at 0.

(ii) $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

Show that :

a) $f(0) = 0$;

b) f is continuous everywhere.

proof :

a) Putting $x = y = 0$,

$$f(0+0) = f(0) + f(0)$$

$$f(0) = 2f(0)$$

$$f(0) = 0$$

b) f is continuous at 0 $\Rightarrow \lim_{h \to 0} f(0+h) = f(0)$

$\Rightarrow \lim_{h \to 0} f(h) = f(0) = 0$

Let $x_0 \in \mathbb{R}$,

$\lim_{h \to 0} f(x_0 + h) = \lim_{h \to 0} [f(x_0) + f(h)]$ (Property of f)

$= f(x_0) + \lim_{h \to 0} f(h)$

$= f(x_0)$

$\therefore f$ is continuous everywhere.

4.3 Sequential Criterion for Continuity

Theorem 4.3.1

A function f is continuous at $x = c$ if and only if for every sequence $\{a_n\}$ with $a_n \neq c$ $\forall n \in \mathbb{N}$ and $\lim_{n \to \infty} a_n = c$, we have $\lim_{n \to \infty} f(a_n) = f(\lim_{n \to \infty} a_n) = f(c)$.

Example 4.3.1

Think : Find $\lim_{n\to\infty} \sqrt{\frac{n^2+1}{4n^2+3}}$.

How did we do ?

① $\lim_{n\to\infty} \frac{n^2+1}{4n^2+3} = \frac{1}{4}$

② $\lim_{n\to\infty} \sqrt{\frac{n^2+1}{4n^2+3}} \overset{(*)}{=} \sqrt{\lim_{n\to\infty} \frac{n^2+1}{4n^2+3}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

so in general, (*) means $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$, why it is true ?

Consider $a_n = \frac{n^2+1}{4n^2+3}$, we have $\lim_{n\to\infty} a_n = \frac{1}{4}$

Also, we know $f(x) = \sqrt{x}$ is continuous at $\frac{1}{4}$,

$\therefore \lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$

$\lim_{n\to\infty} \sqrt{\frac{n^2+1}{4n^2+3}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

Example 4.3.2

Consider

$$f(x) = \begin{cases} 1 & \text{if } x=0 \\ 0 & \text{if } x\neq 0 \end{cases}, \quad \text{and } a_n = \frac{1}{n}.$$

Note : f is NOT continuous at $x=0$.

$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(\frac{1}{n}) = \lim_{n\to\infty} 0 = 0$

$f(\lim_{n\to\infty} a_n) = f(0) = 1$

$\therefore \lim_{n\to\infty} f(a_n) \neq f(\lim_{n\to\infty} a_n)$

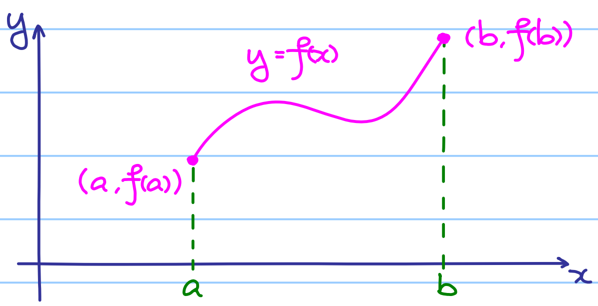
4.4 Continuous on $[a,b]$

Definition 4.4.1

Let $f:[a,b]\longrightarrow\mathbb{R}$

f is said to be continuous at $x=a$ if $\lim\limits_{x\to a^+} f(x)=f(a)$;

f is said to be continuous at $x=b$ if $\lim\limits_{x\to b^-} f(x)=f(b)$.



(We cannot talk about $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to b^+} f(x)$!)

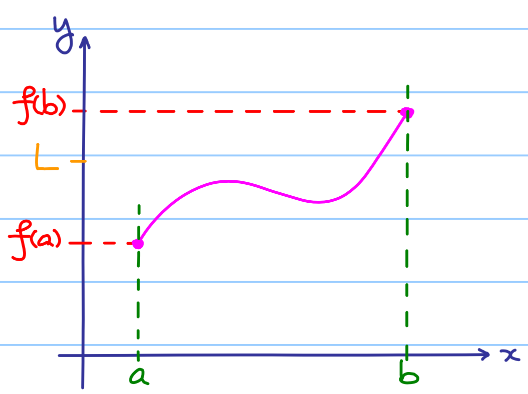
Furthermore, if a function $f:[a,b]\to\mathbb{R}$ is continuous at every point $x\in[a,b]$, then f is said to be continuous on $[a,b]$.

Theorem 4.4.1 (Intermediate Value Theorem)

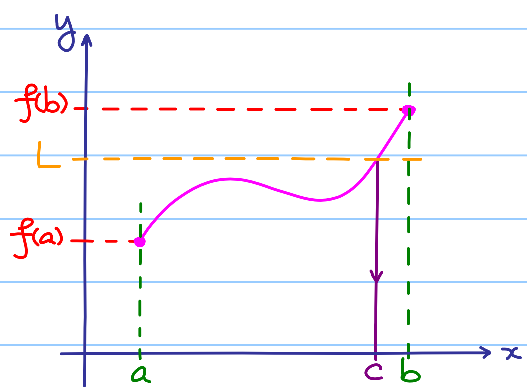
Suppose that f is continuous on $[a,b]$ and $f(a)<f(b)$.

Furthermore, if $L\in\mathbb{R}$ such that $f(a)<L<f(b)$,

then there exists (at least one) $c\in(a,b)$ such that $f(c)=L$.



$\Rightarrow$



Similar result holds for $f(a)>L>f(b)$. (What is the picture?)

Example 4.4.1

Let $f(x) = x^2$

① $f(1) = 1 < 2 < 4 = f(2)$

② f is continuous on $[1,2]$ (In fact, on $\mathbb{R}$)

By Intermediate Value Theorem, there exists $c \in (1,2)$ such that $f(c) = c^2 = 2$.

c is $\sqrt{2}$ by definition!

$\therefore 1 < \sqrt{2} < 2$ (estimates $\sqrt{2}$)

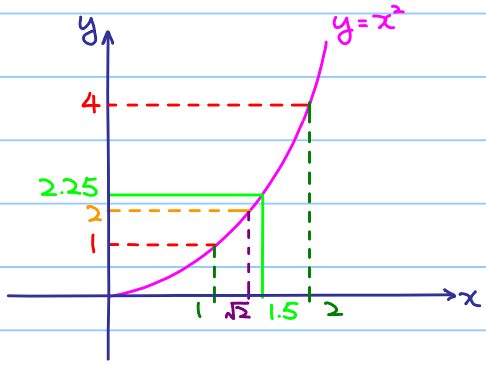
We can further obtain a better estimation by:

① Take the mid-point of $[1,2]$, i.e. 1.5.

② $f(1.5) = 2.25 > 2$.

③ $f(1) = 1 < 2 < 2.25 = f(1.5)$

$\therefore 1 < \sqrt{2} < 1.5$



Repeating again and again to obtain better and better estimation.

It is well-known as method of bisection!

Example 4.4.2

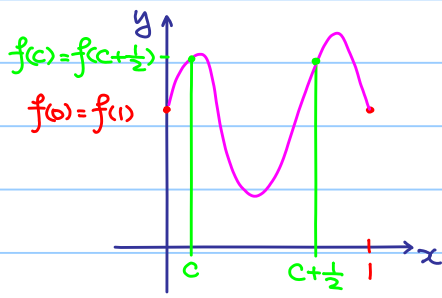
Let $f: [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$.

Prove that there exists $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c+\frac{1}{2})$.

Let $g(x) = f(x) - f(x+\frac{1}{2})$ which is continuous on $[a,b]$

$g(0) = f(0) - f(\frac{1}{2})$

$g(\frac{1}{2}) = f(\frac{1}{2}) - f(1) = -g(0)$



There are 3 cases:

① $g(0) = 0$, done! (Take $c = 0$)

② $g(0) > 0$, then $g(\frac{1}{2}) < 0$

③ $g(0) < 0$, then $g(\frac{1}{2}) > 0$

Intermediate Value Theorem

$\Rightarrow \exists\, c \in (0, \frac{1}{2})$ s.t. $g(c) = 0$

$f(c) - f(c+\frac{1}{2}) = 0$

i.e. $f(c) = f(c+\frac{1}{2})$

Example 4.4.3

Let $f(x)=x^3+bx^2+cx+d$ where $b,c,d\in\mathbb{R}$.

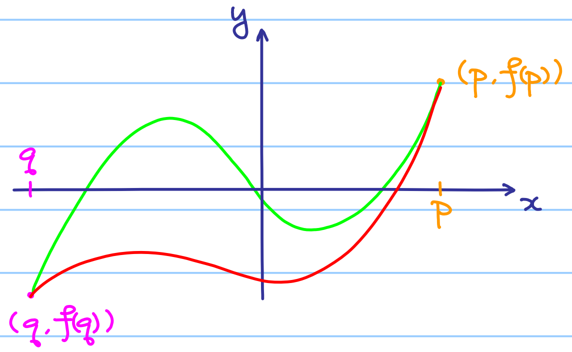
Prove that the equation $f(x)=0$ has at least one real root.

$$f(x)=x^3+bx^2+cx+d$$
$$=x^3\left(1+\frac{b}{x}+\frac{c}{x^2}+\frac{d}{x^3}\right)$$

We can choose $p>0$ such that if $x=p$, $1+\frac{b}{p}+\frac{c}{p^2}+\frac{d}{p^3}>0$

Similarly, we can choose $q<0$ such that if $x=q$, $1+\frac{b}{q}+\frac{c}{q^2}+\frac{d}{q^3}>0$

Then $f(q)<0<f(p)$.



What is the graph of $y=f(x)$?

Red? Green?

Anyway, they cut the x-axis!

f is continuous on $[q,p]$.

$\therefore$ By Intermediate Value Theorem, there exists $x_0\in(q,p)$ such that $f(x_0)=0$

Remark:

1) By factor theorem, $(x-x_0)$ is a factor of $f(x)$.

2) This idea can be generalized to any polynomial $f(x)$ with odd degree.

4.5 Relative and Absolute Extrema

Definition 4.5.1

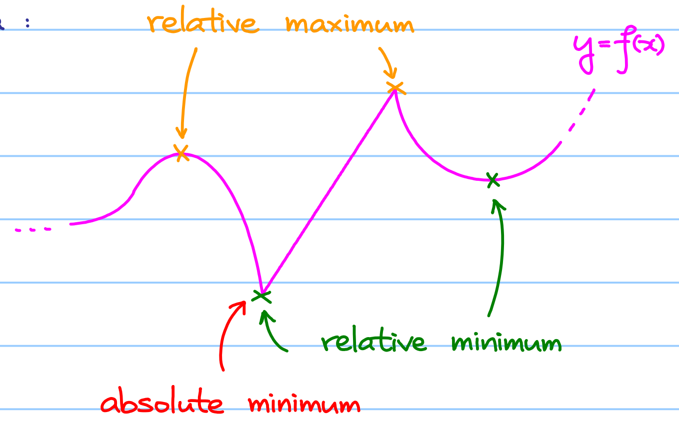
f has an absolute maximum (resp. minimum) point at a if

$f(x)\le f(a)$ (resp. $f(x)\ge f(a)$) for all x in the domain of f.

f has a relative maximum (resp. minimum) point at a if

$f(x)\le f(a)$ (resp. $f(x)\ge f(a)$) for all x in a neighborhood of a.

Idea :



Note : No absolute maximum in this case .

Remark :

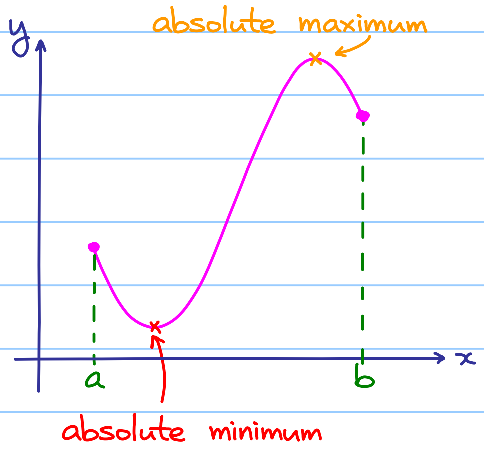
1) We simply use maximum / minimum to refer relative maximum / minimum .

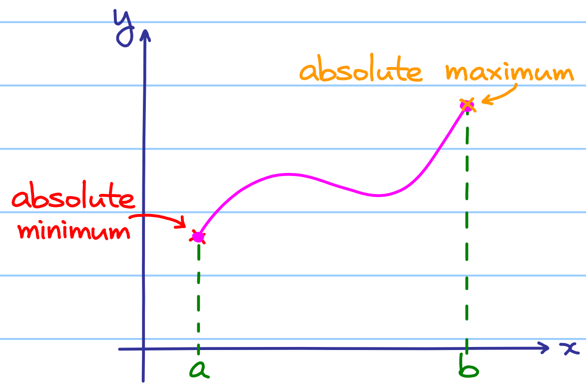
2) Absolute maximum / minimum are also called global maximum / minimum .

Theorem 4.5.1 (Maximum-Minimum Theorem)

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a continuous function .

Then f has an absolute maximum and an absolute minimum on $[a,b]$.





Absolute maximum / minimum may be attained at the boundary points of $[a,b]$.

Main question : Given a function , how to find all absolute / relative extrema ?

Differentiation provides a powerful tool for that .